

Equivalence of Lower Bounds on the Number of Perfect Pairs

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December 23, 2014

Abstract

Let $c(\mathcal{F})$ be the number of perfect pairs of \mathcal{F} and $c(G)$ be the maximum of $c(\mathcal{F})$ over all (near-) one-factorizations \mathcal{F} of G . Wagner showed that for odd n , $c(K_n) \geq \frac{n * \phi(n)}{2}$ and for m and n which are odd and co-prime to each other, $c(K_{mn}) \geq 2 * c(K_m) * c(K_n)$. In this note, we establish that both these results are equivalent in the sense that they both give rise to the same lower bound.

Keywords: complete graph, equivalence of lower bounds, near-one-factorization, near-one-factor of a product graph, number of perfect pairs

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1 Introduction

A one-factor of a graph G of even order is a set of edges that cover each vertex exactly once. In other words, it is a regular spanning sub-graph of degree one [25, 14]. A one-factorization of G is a partition of the edge set into a set of one-factors [17, 25, 10]. Analogously, a near-one-factor of a graph $G = (V, E)$ of odd order is a one-factor of $G \setminus v$ for some $v \in V$, and a near-one-factorization of G is a partition of E into near-one-factors. Our focus in this note is on near-one-factors and near-one-factorizations.

A one-factorization \mathcal{F} of a complete graph K_{2n} on $2n$ vertices consists of $2n - 1$ one-factors $F_1, F_2, \dots, F_{2n-1}$. A near-one-factorization \mathcal{F} of K_{2n-1} also consists of $2n - 1$ near-one-factors $F_1, F_2, \dots, F_{2n-1}$.

A pair of one-factors F_k and F_j in a one-factorization is said to be perfect if $F_k \cup F_j$ induces a Hamiltonian cycle in G [4]. A pair of near-one-factors is called perfect if their union is a Hamiltonian path of G . If every pair of (near-) one-factors of a (near-) one-factorization is perfect then the (near-) one-factorization is called perfect.

Define $c(\mathcal{F})$ to be the number of perfect pairs of \mathcal{F} and $c(G)$ to be the maximum of $c(\mathcal{F})$ over all (near-) one-factorizations \mathcal{F} of G [22, 23, 24]. Perfect one-factorization conjecture says that for $m \geq 2$, $c(K_{2m}) = \binom{2m-1}{2}$ [3, 5, 7, 12, 15, 18]. This conjecture is still open except for the case when m is prime or $2m - 1$ is prime or $2m \in \{16, 28, 36, 40, 50, 52, 126, 170, 244, 344, 730, 1332, 1370, 1850, 2198, 3126, 6860, 12168, 16808, 29792\}$ [1, 2, 6, 8, 9, 11, 13, 16, 19, 20, 21, 22, 23, 24, 26, 27]. It can be readily argued that a complete graph of order $2m$ has a perfect one-factorization if and only if a complete graph of order $2m - 1$ has a perfect near-one-factorization.

As part of an attempt to prove the perfect one-factorization conjecture, Wagner, in [22], shows that for odd n , $c(K_n) \geq n * \phi(n)/2$, where $\phi(n)$ is the Euler's totient function. Also proven in the same paper is that $c(K_{mn}) \geq 2 * c(K_m) * c(K_n)$ if m and n are odd and are relatively prime. Though the later result can be used with other relevant information to arrive at a better lower bound but it is equivalent to the former result.

In this note, we show that the two results mentioned above are equivalent in the sense that they both give rise to the same lower bound. This equivalence is established by coming up with a one-to-one correspondence between both the sets of near-one-factors.

Notation 1.1 $\frac{k}{r} \bmod n$ denotes $(k * \text{multiplicative inverse of } r \text{ with respect to } n) \bmod n$, if the multiplicative inverse of r with respect to n exists.

1.1 Our Results

Main contribution of the paper is that the two results, namely Proposition 2 and Theorem 3 of [22], are equivalent in the sense that they both give rise to the same lower bound. It also extends the definition of a near-one-factor given in Proposition 2 of [22] to one-factors and comes up with an alternative treatment for the proposition. In addition, it renders an algebraic description to the construction of a near-one-factor of a product graph from those of its constituent graphs and supplies Theorem 3 of [22] with an algebraic proof.

1.2 Organization of the Paper

Section 2 examines the definition of a near-one-factor given in the proof of Proposition 2 of [22], extends it to one-factors with suitable modifications, and provides an alternative treatment to Proposition 2. Section

3 proposes an algebraic description to the construction of a near-one-factor of a product graph from those of its constituent graphs and supplies Theorem 3 of [22] with an algebraic proof. Section 4 shows that both Proposition 2 and Theorem 3 of [22] are equivalent in the sense that they both give rise to the same lower bound. This is achieved by establishing a one-to-one correspondence between the set of near-one-factors of the product graph and the set of products of near-one-factors of the constituent graphs. Concluding remarks are in Section 5.

2 One-Factors and Perfect Pairs

This section examines the definition of a one-factor given in [22] and explores the conditions under which two of them form a perfect pair.

Consider a graph F_k , $k \in \{0, 1, 2, \dots, n-1\}$ on n vertices with adjacency matrix A_k that has 1 as its i, j^{th} element, where $i \neq j$ and $i + j = k \pmod n$.

Claim 2.1 *Let n be odd. Then F_k , $k \in \{0, 1, 2, \dots, n-1\}$ defined as above is a near-one-factor of K_n , whose isolated vertex is $\frac{k}{2} \pmod n$.*

Proof. Fix k . Then for every $i \in \{0, 1, 2, \dots, n-1\}$ there is a $j \in \{0, 1, 2, \dots, n-1\}$ such that $i + j = k \pmod n$. This is because the set $\{0, 1, 2, \dots, n-1\}$ is closed and each element has additive inverse in the set with modulo n addition as the operation. Also $i \neq j$ except for $i = j = \frac{k}{2} \pmod n$; for if $i = j$, then $2i = k \pmod n$, which implies $i = \frac{k}{2} \pmod n$. $\frac{k}{2} \pmod n$ is unique because n is odd and hence multiplicative inverse of 2 exists. Moreover, the pair i and j is unique in the sense that for a given k and i there is a unique j with this property. Claim follows because the first part of the discussion implies that every vertex of K_n occurs in F_k and the last statement implies that each vertex occurs exactly once. \square

Remark 2.2 *When n is even, only the graphs F_k , $k \in \{1, 3, \dots, n-1\}$ are the one-factors of K_n . For the graph F_k , k even, to be a one-factor of K_n , the adjacency matrix A_k should be such that has 1 in its i, j^{th} element, where $i \neq j$ and either*

1. $i, j \in \{\frac{k}{2}, \frac{n+k}{2}\}$ or
2. $i, j \notin \{\frac{k}{2}, \frac{n+k}{2}\}$, and $i + j \equiv k \pmod n$

Following lemma aids in arriving at other results.

Lemma 2.3 *Let n be odd. Also, let F_k and F_ℓ be two near-one-factors of K_n defined as above. Then the i^{th} edge of the union of these two near-one-factors, starting from the isolated vertex of the near-one-factor F_k , is either*

$$\left(\left(\frac{ik}{2} - \frac{(i-1)\ell}{2}\right) \pmod n, \left(\frac{(i+1)\ell}{2} - \frac{ik}{2}\right) \pmod n\right), \text{ if } i \text{ is odd}$$

or

$$\left(\left(\frac{i\ell}{2} - \frac{(i-1)k}{2}\right) \pmod n, \left(\frac{(i+1)k}{2} - \frac{i\ell}{2}\right) \pmod n\right), \text{ if } i \text{ is even}.$$

Proof. The other vertex of the edge of F_ℓ connecting the isolated vertex of F_k , i.e. $\frac{k}{2} \bmod n$, is $(\ell - \frac{k}{2}) \bmod n$. So, the first edge of the union starting from the isolated vertex of F_k is $(\frac{k}{2} \bmod n, (\ell - \frac{k}{2}) \bmod n)$. Similarly, the other vertex of the edge of F_k connecting the vertex $(\ell - \frac{k}{2}) \bmod n$ is $(\frac{3k}{2} - \ell) \bmod n$. So, the second edge of the union starting from the isolated vertex of F_k is $((\ell - \frac{k}{2}) \bmod n, (\frac{3k}{2} - \ell) \bmod n)$. Continuing in this way we have the third edge as $((\frac{3k}{2} - \ell) \bmod n, (2\ell - \frac{3k}{2}) \bmod n)$, fourth edge as $((2\ell - \frac{3k}{2}) \bmod n, (\frac{5k}{2} - 2\ell) \bmod n)$, etc. In general, the i^{th} edge of the union is either

$$((\frac{ik}{2} - \frac{(i-1)\ell}{2}) \bmod n, (\frac{(i+1)\ell}{2} - \frac{ik}{2}) \bmod n) \text{ for odd } i$$

or

$$((\frac{i\ell}{2} - \frac{(i-1)k}{2}) \bmod n, (\frac{(i+1)k}{2} - \frac{i\ell}{2}) \bmod n) \text{ for even } i.$$

□

Lemma 2.4 *Let n be odd. Also, let F_k and F_ℓ be two near-one-factors of K_n defined as above. Then the path starting from the isolated vertex of either of the near-one-factors in the union of these two does not contain a cycle if and only if $(k - \ell)$ is relatively prime to n .*

Proof. With out loss of generality, let us assume that the starting vertex of the path is the isolated vertex of F_k . Then from the previous lemma (Lemma 2.3) the i^{th} edge of the path is either

$$((\frac{ik}{2} - \frac{(i-1)\ell}{2}) \bmod n, (\frac{(i+1)\ell}{2} - \frac{ik}{2}) \bmod n) \text{ if } i \text{ is odd}$$

or

$$((\frac{i\ell}{2} - \frac{(i-1)k}{2}) \bmod n, (\frac{(i+1)k}{2} - \frac{i\ell}{2}) \bmod n) \text{ if } i \text{ is even}.$$

For there to be a cycle on this path, there must exist two distinct positive integers i and j such that either

$$(\frac{ik}{2} - \frac{(i-1)\ell}{2}) \bmod n = (\frac{jk}{2} - \frac{(j-1)\ell}{2}) \bmod n$$

or

$$(\frac{i\ell}{2} - \frac{(i-1)k}{2}) \bmod n = (\frac{j\ell}{2} - \frac{(j-1)k}{2}) \bmod n$$

But they are equivalent to

$$(\frac{i-j}{2})(k-\ell) \equiv 0 \bmod n$$

$$\Leftrightarrow (i-j)(k-\ell) \equiv 0 \bmod n \quad (\because \text{multiplicative inverse of } 2 \text{ with respect to } n \text{ exists})$$

$$\Leftrightarrow i = j \quad (\because (k-\ell) \text{ is relatively prime to } n \text{ and both } i \text{ and } j \text{ are less than } n)$$

□

Lemma 2.5 *Let n be odd. Then two near-one-factors F_k and F_ℓ defined as above is a perfect pair if and only if $k - \ell$ is relatively prime to n .*

Proof. Consider the path starting from the isolated vertex of F_k in the union of the two near-one-factors. As there is no cycle on this path, it follows that the other end of the path must be the isolated vertex $\frac{\ell}{2} \bmod n$

of F_ℓ . So,

$$\begin{aligned}
& \frac{\ell}{2} \bmod n = \left(\frac{(i+1)k}{2} - \frac{i\ell}{2} \right) \bmod n \\
\Leftrightarrow & \frac{\ell}{2} + \frac{i\ell}{2} - \frac{(i+1)k}{2} \equiv 0 \bmod n \\
\Leftrightarrow & \frac{(i+1)}{2}(\ell - k) \equiv 0 \bmod n \\
\Leftrightarrow & (i+1)(\ell - k) \equiv 0 \bmod n \quad (\because 2 \text{ has multiplicative inverse}) \\
\Leftrightarrow & (i+1) \text{ is a multiple of } n \quad (\because k - \ell \text{ is relatively prime to } n) \\
\Leftrightarrow & i+1 = n \quad (\because \text{length of any path in the union of two near-one-factors is less than } n) \\
\Leftrightarrow & i = n - 1.
\end{aligned}$$

That is the path connecting the isolated vertices is a hamiltonian path. Hence the claim. \square

Remark 2.6 From the discussion above it follows that a near-one-factor F_k forms a perfect pair with another near-one-factor F_ℓ if and only if $k - \ell$ is relatively prime to n . For a fixed k , the number of such ℓ 's is equal to $\phi(n)$. (This may be proved by observing that a pair of integers ℓ and $c * n - \ell$, c is some integer, is either both relatively prime to n or both not.) Hence the number of perfect pairs with one of the near-one-factor in the pair being F_k is $\phi(n)$. So, the total number of perfect pairs is $\frac{(n * \phi(n))}{2}$. Note that this is the result of the Proposition 2 of [22]).

3 One-Factors and Product Graphs

This section provides an algebraic description of the construction of a near-one-factor of a product graph from those of its constituent graphs. It also analyzes the conditions under which two near-one-factors of a product graph form a perfect pair and supplements Theorem 3 of [22] with an algebraic proof.

Definition 3.1 [Product Graph] The product graph $M \times N$ of its constituent graphs M and N is defined as follows:

1. Vertex set, $V(M \times N)$, of the product graph $M \times N$ is the cartesian product of the vertex sets $V(M)$ and $V(N)$ of its constituent graphs M and N respectively. That is $V(M \times N) = V(M) \times V(N)$.
2. The edge set of the product graph, $E(M \times N)$, is $\{(v, w), (v', w')\} \in E(M \times N)$ if and only if $\{v, v'\} \in E(M)$ and $\{w, w'\} \in E(N)$.

Claim 3.2 Let s and t be odd positive integers. Also let $0 \leq i, i', k < s$ and $0 \leq j, j', \ell < t$. Further, let G_k and H_ℓ denote near-one-factors of the complete graphs K_s and K_t respectively. Define $D_{k,\ell}$ to be a square $0, 1$ matrix whose rows and columns are indexed by ordered pairs (i, j) such that the element in $(i, j)^{th}$ row and $(i', j')^{th}$ column is 1 if and only if $(i, j) \neq (i', j')$, $(i + i') \bmod s = k$, and $(j + j') \bmod t = \ell$. Then $D_{k,\ell}$ is an adjacency matrix of the product graph $G_k \times H_\ell$.

Proof. We have the vertex set, $V(G_k \times H_\ell)$, of the product graph $G_k \times H_\ell$ as $\{(i, j) : i \in V(G_k) \text{ and } j \in V(H_\ell)\}$. Also, $\{(i, j), (i', j')\}$ is an edge in the product graph if and only if both $\{i, i'\}$ and $\{j, j'\}$ are edges in the graphs G_k and H_ℓ respectively. But $\{i, i'\}$ and $\{j, j'\}$ are edges in their respective graphs if and only if $i \neq i', j \neq j', (i + i') \bmod s = k$, and $(j + j') \bmod t = \ell$. Hence the claim. \square

Claim 3.3 Let $n = s \times t$. Then $D_{k,\ell}$, where $0 \leq k < s$ and $0 \leq \ell < t$, defined in Claim 3.2 is an adjacency matrix of a near-one-factor of K_n , whose isolated vertex is $((\frac{k}{2}) \bmod s, (\frac{\ell}{2}) \bmod t)$. That is, the product graph $G_k \times H_\ell$ is a near-one-factor of the complete graph K_n .

Proof. Fix k and ℓ and argue as in Claim 2.1 by treating k as the ordered pair (k, ℓ) , i as (i, j) , j as (i', j') , and n as (s, t) . \square

Remark 3.4 Let $G_k, G_{k'}$ denote a pair of near-one-factors of the complete graph K_s and $H_\ell, H_{\ell'}$ denote a pair of near-one-factors of the complete graph K_t . From the previous claim we have $G_p \times H_q$, $p \in \{k, k'\}$ and $q \in \{\ell, \ell'\}$, are near-one-factors of the complete graph K_{st} .

Claim 3.5 Let $G_k, G_{k'}$, H_ℓ , and $H_{\ell'}$ be as defined in the above remark. Then a pair of near-one-factors from the set $\{G_p \times H_q : p \in \{k, k'\} \text{ and } q \in \{\ell, \ell'\}\}$ of the complete graph K_{st} is perfect if and only if both the pairs, namely $G_k, G_{k'}$ and $H_\ell, H_{\ell'}$, are perfect for the corresponding complete graphs.

Proof. Consider two near-one-factors $G_k \times H_\ell$ and $G_{k'} \times H_{\ell'}$ of the complete graph K_{st} . Arguing as in Section 2, these two near-one-factors can be shown to form a perfect pair if and only if $(k - k')$ is relatively prime to s and $(\ell - \ell')$ is relatively prime to t . But from Lemma 2.5, it means that both the pairs namely $G_k, G_{k'}$ and $H_\ell, H_{\ell'}$ are perfect. Hence the claim. \square

Remark 3.6 Since $\gcd((x - x), s) = s$, it follows from the proof of the above claim that there are two perfect pairs, namely $G_k \times H_\ell, G_{k'} \times H_{\ell'}$ and $G_{k'} \times H_\ell, G_k \times H_{\ell'}$ of K_{st} for every perfect pair $G_k, G_{k'}$ of K_s and perfect pair $H_\ell, H_{\ell'}$ of K_t . So, the number of perfect pairs of K_{st} is more than or equal to twice the number of perfect pairs of K_s times the number of perfect pairs of K_t . That is, $c(K_{st}) \geq 2 * c(K_s) * c(K_t)$. Note that this is the result of Theorem 3 of [22].

4 Equivalence of Lower Bounds

In the previous section, we have shown that the product of near-one-factors of constituent graphs is a near-one-factor of the product graph. We now show that, under certain mild conditions, a near-one-factor of a product graph is the product of near-one-factors of its constituent graphs. This is achieved by establishing a one-to-one correspondence between the set of near-one-factors of the product graph and the set of products of near-one-factors of its constituent graphs. So, it implies that the lower bounds obtained in Sections 2 and 3 are equivalent.

Lemma 4.1 Let s and t be odd positive integers and are co-prime to each other. Also, let $n = s \times t$. Further, let A_p , $0 \leq p < n$, and $D_{k,\ell}$, $0 \leq k < s$, $0 \leq \ell < t$, be 0, 1 matrices defined in Section 2 and Claim 3.2 respectively. Then there is a one-to-one correspondence between the sets $\{A_p : 0 \leq p < n\}$ and $\{D_{k,\ell} : 0 \leq k < s, 0 \leq \ell < t\}$.

Proof. Define a mapping from the set of matrices $\{A_p : 0 \leq p < n\}$ to the set of matrices $\{D_{k,\ell} : 0 \leq k < s, 0 \leq \ell < t\}$ such that the matrix A_p gets mapped to $D_{k,\ell}$ if and only if $p \bmod s = k$ and $p \bmod t = \ell$. Since s and t are co-prime to each other, by Chinese Remainder Theorem, this mapping is a one-to-one correspondence. \square

Remark 4.2 Since $\{A_p : 0 \leq p < n\}$ denote the adjacency matrices of a set of near-one-factors of K_n and $\{D_{k,\ell} : 0 \leq k < s, 0 \leq \ell < t\}$ also denote the adjacency matrices of another set of near-one-factors of K_n , it follows from the above lemma that these two sets of near-one-factors are one and the same. So, the number of perfect pairs in both the sets of near-one-factors is same.

5 Conclusions

This note establishes that the apparently two different lower bounds derived in [22] are one and the same. It also extends the definition of a near-one-factor given in Proposition 2 of [22] to one-factors and comes up with an alternative treatment for the proposition. In addition, it renders an algebraic description to the construction of a near-one-factor of a product graph from those of its constituent graphs and supplies Theorem 3 of [22] with an algebraic proof.

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